

# A CATEGORY OF MULTIPLIER BIMONOIDS

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ABSTRACT. The central object studied in this paper is a *multiplier bimonoid in a braided monoidal category*  $\mathcal{C}$ , introduced and studied in [4]. Adapting the philosophy in [6], and making some mild assumptions on the category  $\mathcal{C}$ , we consider a category  $\mathcal{M}$  whose objects are certain semigroups in  $\mathcal{C}$  and whose morphisms  $A \rightarrow B$  can be regarded as suitable multiplicative morphisms from  $A$  to the multiplier monoid of  $B$ . We equip this category  $\mathcal{M}$  with a monoidal structure and describe multiplier bimonoids in  $\mathcal{C}$  (whose structure morphisms belong to a distinguished class of regular epimorphisms) as certain comonoids in  $\mathcal{M}$ . This provides us with one possible notion of morphism between such multiplier bimonoids.

## 1. INTRODUCTION

A bialgebra over a field or, more generally, a bimonoid in a braided monoidal category, is an object carrying a monoid and a comonoid structure subject to compatibility conditions that can be interpreted as saying that a bimonoid is a monoid in the category of comonoids; equivalently, it is a comonoid in the category of monoids.

A multiplier bialgebra over a field [3] or, more generally, a multiplier bimonoid in a braided monoidal category [4], is a generalization which is no longer a monoid or a comonoid in the base category. However, Janssen and Vercruysse constructed in [6] a monoidal category, whose objects are certain non-unital algebras (say over a field), and in which the comonoids include the multiplier Hopf algebras of Van Daele [7].

Our aim in this paper is to generalize and strengthen this result. Namely, under mild assumptions (involving a class  $\mathcal{Q}$  of regular epimorphisms) we construct a category  $\mathcal{M}$  of certain semigroups in a braided monoidal category  $\mathcal{C}$ . We describe multiplier bimonoids in  $\mathcal{C}$  (whose structure morphisms lie in  $\mathcal{Q}$ ) as certain comonoids in  $\mathcal{M}$ . Defining the morphisms between such multiplier bimonoids as the morphisms between the corresponding comonoids in  $\mathcal{M}$ , we obtain a category of multiplier bimonoids in  $\mathcal{C}$ .

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## 2. MULTIPLIER MONOIDS AND THEIR MORPHISMS

We begin by describing what is meant by multiplier monoids in closed braided monoidal categories and we characterize multiplicative morphisms with codomain a multiplier monoid.

**2.1. Assumptions on the base category.** Throughout, we work in a braided monoidal category  $\mathcal{C}$ . The composition of morphisms  $f: A \rightarrow B$  and  $g: B \rightarrow C$  will be denoted by  $g.f: A \rightarrow C$ . The monoidal product of  $A$  and  $B$  will be denoted by  $AB$ , the monoidal unit by  $I$  and the braiding by  $c$ . For  $n$  copies of the same object  $A$ , we also use the power notation  $AA \dots A = A^n$ .

We fix a class  $\mathcal{Q}$  of regular epimorphisms in  $\mathcal{C}$  which is closed under composition and monoidal product, contains the isomorphisms, and is right-cancellative in the sense that if  $s: A \rightarrow B$  and  $t.s: A \rightarrow C$  are in  $\mathcal{Q}$ , then so is  $t: B \rightarrow C$ . Since each  $q \in \mathcal{Q}$  is a regular epimorphism, it is the coequalizer of some pair of maps. Finally we suppose that this pair may be chosen in such a way that the coequalizer is preserved by taking the monoidal product with any object.

These assumptions are always satisfied when  $\mathcal{Q}$  consists of the split epimorphisms. In the other main case  $\mathcal{Q}$  consists of the regular epimorphisms. In this case, we need to suppose that the regular epimorphisms are closed under composition, as is the case in any regular category; we also need to suppose that (enough) coequalizers are preserved by taking the monoidal product with a fixed object, as will be true if the monoidal category is *closed* (see Paragraph 2.3 below). In particular, we may take  $\mathcal{Q}$  to be all the regular epimorphisms if  $\mathcal{C}$  is the symmetric monoidal category of modules over a commutative ring.

**2.2. Semigroups with non-degenerate multiplication.** By a *semigroup* in the braided monoidal category  $\mathcal{C}$  we mean a pair  $(A, m)$  consisting of an object  $A$  of  $\mathcal{C}$  and a morphism  $m: A^2 \rightarrow A$  – called the *multiplication* – which obeys the associativity condition  $m.m1 = m.1m$ . If the semigroup has a *unit* – that is, a morphism  $u: I \rightarrow A$  such that  $m.u1 = 1 = m.1u$  – then we say that  $A$  is a *monoid*.

The multiplication – or, alternatively, the semigroup  $A$  – is said to be *non-degenerate* if for any objects  $X, Y$  of  $\mathcal{C}$ , both maps

$$\begin{aligned} \mathcal{C}(X, YA) &\rightarrow \mathcal{C}(XA, YA), & f &\mapsto XA \xrightarrow{f1} YA^2 \xrightarrow{1m} YA & \text{and} \\ \mathcal{C}(X, AY) &\rightarrow \mathcal{C}(AX, AY), & g &\mapsto AX \xrightarrow{1g} A^2Y \xrightarrow{m1} AY \end{aligned}$$

are injective. The multiplication of a monoid is always non-degenerate. (Requiring injectivity of these maps for *any* object  $Y$  is quite strong and can often be avoided. For a careful analysis of the class of objects  $Y$  with this property consult [2].)

**2.3. Closed monoidal categories.** The braided monoidal category  $\mathcal{C}$  is said to be *closed* if for each object  $X$  the functor  $X(-): \mathcal{C} \rightarrow \mathcal{C}$  possesses a right adjoint (equivalently, if each  $(-)X$  possesses a right adjoint). We write  $[X, -]$  for the right adjoint; then the components of the unit and the counit have the form

$$Y \xrightarrow{\eta} [X, XY], \quad X[X, Y] \xrightarrow{\epsilon} Y.$$

The right adjoint  $[X, -]$  is functorial in the variable  $X$ , so that in fact we have a functor  $[-, -]: \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$  and now the adjointness

$$\mathcal{C}(XY, Z) \cong \mathcal{C}(Y, [X, Z])$$

is natural in all three variables.

**Lemma 2.4.** *For a semigroup  $A$  in a closed braided monoidal category  $\mathcal{C}$ , the following assertions are equivalent.*

- (i) *The multiplication  $m: A^2 \rightarrow A$  is non-degenerate.*
- (ii) *For any object  $Y$ , both morphisms*

$$\begin{aligned} r_Y &:= AY \xrightarrow{\eta} [A, A^2Y] \xrightarrow{[A, m1]} [A, AY] \quad \text{and} \\ l_Y &:= YA \xrightarrow{\eta} [A, AY A] \xrightarrow{[A, c1]} [A, Y A^2] \xrightarrow{[A, 1c]} [A, Y A^2] \xrightarrow{[A, 1m]} [A, Y A] \end{aligned}$$

*are monomorphisms.*

*Proof.* Since  $r_Y$  and  $m1: A^2Y \rightarrow AY$  are mates under the adjunction  $A(-) \dashv [A, -]$ , the equality  $r_Y.f = r_Y.g$  holds for any morphisms  $f$  and  $g: X \rightarrow AY$  if and only if  $m1.1f = m1.1g$ . Symmetrically, since  $l_Y$  and  $1m.c_{A,YA}: AY A \rightarrow YA$  are mates, the equality  $l_Y.f = l_Y.g$  holds for any  $f, g: X \rightarrow YA$  if and only if  $1m.f1 = 1m.g1$ .  $\square$

**2.5.  $\mathbb{M}$ -morphisms.** For a monoid  $B = (B, m, u)$  and an object  $A$ , to give a morphism  $f: A \rightarrow B$  in  $\mathcal{C}$  is equivalently to give a morphism  $f_1: AB \rightarrow B$  compatible with the right actions of  $B$ , in the sense that the first diagram in (2.1) below commutes

$$\begin{array}{ccc} AB^2 & \xrightarrow{1m} & AB \\ f_1 1 \downarrow & & \downarrow f_1 \\ B^2 & \xrightarrow{m} & B \end{array} \quad \begin{array}{ccc} B^2 A & \xrightarrow{m1} & BA \\ 1 f_2 \downarrow & & \downarrow f_2 \\ B^2 & \xrightarrow{m} & B. \end{array} \quad (2.1)$$

Under this bijection  $f_1$  is given by  $m.f1$ , and  $f$  given by  $f_1.1u$ . Dually, it is equivalent to giving a morphism  $f_2: BA \rightarrow B$  compatible with the left actions, in the sense that the second diagram in (2.1) commutes. Furthermore, the resulting  $f_1$  and  $f_2$  are related by commutativity of the diagram

$$\begin{array}{ccc} BAB & \xrightarrow{1f_1} & B^2 \\ f_2 1 \downarrow & & \downarrow m \\ B^2 & \xrightarrow{m} & B. \end{array} \quad (2.2)$$

If now  $A$  is a semigroup, then  $f: A \rightarrow B$  will be a semigroup morphism if and only if the diagrams

$$\begin{array}{ccc} A^2 B & \xrightarrow{1f_1} & AB \\ m1 \downarrow & & \downarrow f_1 \\ AB & \xrightarrow{f_1} & B \end{array} \quad \begin{array}{ccc} BA^2 & \xrightarrow{f_2 1} & BA \\ 1m \downarrow & & \downarrow f_2 \\ BA & \xrightarrow{f_2} & B \end{array} \quad (2.3)$$

commute.

This motivates the following notion of morphism when  $B$  is just a non-degenerate semigroup.

**Definition 2.6.** If  $A$  is an object and  $B$  is a non-degenerate semigroup in  $\mathcal{C}$ , an  $\mathbb{M}$ -morphism  $f$  from  $A$  to  $B$  is a pair  $(f_1, f_2)$  of morphisms in  $\mathcal{C}$  making the diagram (2.2) commute. We call  $f_1$  and  $f_2$  the *components* of  $f$ , and we represent the  $\mathbb{M}$ -morphism as  $f: A \rightharpoonup B$ . If  $A$  is also a semigroup, we say that the  $\mathbb{M}$ -morphism  $f$  is *multiplicative* if the diagrams (2.3) commute.

*Remark 2.7.* By non-degeneracy, for  $\mathbb{M}$ -morphisms  $f$  and  $g$  to be equal, it suffices that either  $f_1 = g_1$  or  $f_2 = g_2$ ; the other equality then follows. Similarly, for an  $\mathbb{M}$ -morphism to be multiplicative it suffices that either of the diagrams in (2.3) commutes; commutativity of the other then follows.

**Lemma 2.8.** *If  $B$  is a non-degenerate semigroup, and  $f: A \rightarrowtail B$  is an  $\mathbb{M}$ -morphism, then the diagrams in (2.1) commute.*

*Proof.* Commutativity of the first diagram in the claim follows by commutativity of both diagrams

$$\begin{array}{ccc}
 BAB^2 & \xrightarrow{1f_1} & B^3 & \xrightarrow{1m} & B^2 \\
 f_{21} \downarrow & (2.2) & m1 \downarrow & (\text{associativity}) & \downarrow m \\
 B^3 & \xrightarrow{m1} & B^2 & \xrightarrow{m} & B
 \end{array}
 \qquad
 \begin{array}{ccc}
 BAB^2 & \xrightarrow{11m} & BAB & \xrightarrow{1f_1} & B^2 \\
 f_{21} \downarrow & & f_{21} \downarrow & (2.2) & \downarrow m \\
 B^3 & \xrightarrow{1m} & B^2 & \xrightarrow{m} & B
 \end{array}$$

and the associativity and non-degeneracy of  $m$ . A symmetric reasoning applies to the second diagram.  $\square$

**2.9. Multiplier monoids.** Suppose that  $X$  is an object of  $\mathcal{C}$  and  $A$  is a non-degenerate semigroup. If  $\mathcal{C}$  is closed, then there is a bijection between morphisms  $f_1: XA \rightarrow A$  and morphisms  $\check{f}_1: X \rightarrow [A, A]$ , and similarly there is a bijection between morphisms  $f_2: AX \rightarrow A$  and morphisms  $\hat{f}_2: X \rightarrow [A, A]$ . Moreover, the morphisms  $f_1$  and  $f_2$  make the diagram (2.2) commute, and so determine an  $\mathbb{M}$ -morphism, just when  $\check{f}_1$  and  $\hat{f}_2$  make the first diagram in

$$\begin{array}{ccc}
 X & \xrightarrow{\check{f}_1} & [A, A] \\
 \hat{f}_2 \downarrow & & \downarrow \varphi \\
 [A, A] & \xrightarrow{\psi} & [A^2, A]
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{M}(A) & \xrightarrow{\check{e}_1} & [A, A] \\
 \hat{e}_2 \downarrow & & \downarrow \varphi \\
 [A, A] & \xrightarrow{\psi} & [A^2, A].
 \end{array}
 \tag{2.4}$$

commute, where  $\varphi$  and  $\psi$  correspond under the adjunction isomorphism  $\mathcal{C}(A^2[A, A], A) \cong \mathcal{C}([A, A], [A^2, A])$  to the morphisms

$$A^2[A, A] \xrightarrow{1\varepsilon} A^2 \xrightarrow{m} A, \qquad A^2[A, A] \xrightarrow{1c} A[A, A]A \xrightarrow{\varepsilon 1} A^2 \xrightarrow{m} A.$$

If the pullback of  $\varphi$  and  $\psi$  exists, as in the second diagram of (2.4), then there is a bijection between  $\mathbb{M}$ -morphisms  $X \rightarrowtail A$ , and morphisms  $X \rightarrow \mathbb{M}(A)$  in  $\mathcal{C}$ .

Under this bijection, the identity morphism  $\mathbb{M}(A) \rightarrow \mathbb{M}(A)$  will correspond to an  $\mathbb{M}$ -morphism  $e: \mathbb{M}(A) \rightarrowtail A$  with components  $e_1: \mathbb{M}(A)A \rightarrow A$  and  $e_2: A\mathbb{M}(A) \rightarrow A$ . The components of a morphism  $f: X \rightarrow \mathbb{M}(A)$  have the form  $f_1 = e_1.f1$  and  $f_2 = e_2.1f$ .

**Proposition 2.10.** *Consider a non-degenerate semigroup  $A$  in a closed braided monoidal category  $\mathcal{C}$ .*

- (i) *If the pullback  $\mathbb{M}(A)$  in (2.4) exists, then it carries the structure of a unital monoid in  $\mathcal{C}$ .*
- (ii) *For another semigroup  $B$ , a morphism  $f: B \rightarrow \mathbb{M}(A)$  in  $\mathcal{C}$  is multiplicative if and only if the corresponding  $\mathbb{M}$ -morphism  $B \rightarrowtail A$  is so.*

*Proof.* (i) Using (2.2) for  $e$  and functoriality of the monoidal product, we see that  $e_1.1e_1 : \mathbb{M}(A)^2 A \rightarrow A$  and  $e_2.e_2 1 : A\mathbb{M}(A)^2 \rightarrow A$  can be regarded as the components of a morphism  $m : \mathbb{M}(A)^2 \rightarrow \mathbb{M}(A)$  rendering commutative

$$\begin{array}{ccccc} \mathbb{M}(A)^2 A & \xrightarrow{1e_1} & \mathbb{M}(A)A & \xrightarrow{e_1} & A & \xleftarrow{e_2} & A\mathbb{M}(A) & \xleftarrow{e_2 1} & A\mathbb{M}(A)^2 \\ m1 \downarrow & & & & \parallel & & & & \downarrow 1m \\ \mathbb{M}(A)A & \xrightarrow{e_1} & A & \xleftarrow{e_2} & A\mathbb{M}(A). \end{array} \quad (2.5)$$

Applying (2.5) and functoriality of the monoidal product, the components of  $m.1m$  and of  $m.m1$  turn out to be equal to the same morphisms  $e_1.1e_1.1e_1$  and  $e_2.e_2 1.e_2 1$ . This proves the associativity of  $m$ .

The identity morphism  $1 : A \rightarrow A$  can be regarded as the first and the second components of a morphism  $u : I \rightarrow \mathbb{M}(A)$  rendering commutative

$$\begin{array}{ccc} & A & \\ u1 \swarrow & \parallel & \searrow 1u \\ \mathbb{M}(A)A & \xrightarrow{e_1} & A & \xleftarrow{e_2} & A\mathbb{M}(A). \end{array} \quad (2.6)$$

By (2.5), (2.6) and functoriality of the monoidal product, the components of both  $m.1u$  and of  $m.u1$  are equal to  $e_1$  and  $e_2$ . This proves that  $u$  is the unit of  $m$ .

(ii) By (2.5), the components of  $m.f f : B^2 \rightarrow \mathbb{M}(A)$  are  $f_1.1f_1$  and  $f_2.f_2 1$ ; while the components of  $f.m : B^2 \rightarrow \mathbb{M}(A)$  are  $f_1.m1$  and  $f_2.1m$ .  $\square$

### 3. A CATEGORY OF SEMIGROUPS

In the previous section we introduced a notion of  $\mathbb{M}$ -morphism for non-degenerate semigroups; we now turn to composition of  $\mathbb{M}$ -morphisms. This does not seem to be possible in general, but we give sufficient conditions under which it is. Once again, we motivate the definition using the unital case. If  $g : B \rightarrow C$  is a monoid morphism, and  $f : A \rightarrow B$  an arbitrary morphism, then the following diagram commutes

$$\begin{array}{ccccc} ABC & \xrightarrow{1g1} & AC^2 & \xrightarrow{1m} & AC \\ f11 \downarrow & & & & \downarrow f1 \\ B^2C & \xrightarrow{1g1} & BC^2 & \xrightarrow{1m} & BC \\ \downarrow m1 & & \downarrow g11 & & \downarrow g1 \\ & & C^3 & \xrightarrow{1m} & C^2 \\ & & \downarrow m1 & & \downarrow m \\ BC & \xrightarrow{g1} & C^2 & \xrightarrow{m} & C \end{array}$$

which can in turn be read as the equality  $(gf)_1.1g_1 = g_1.f_1 1$  using the notation of Paragraph 2.5.

Now suppose that  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are  $\mathbb{M}$ -morphisms with  $g$  multiplicative. We would like to define a composite  $\mathbb{M}$ -morphism  $g \bullet f$  of  $g$  and  $f$  in such a way

that the diagrams

$$\begin{array}{ccc}
 ABC & \xrightarrow{1g_1} & AC \\
 f_1 1 \downarrow & & \downarrow (g \bullet f)_1 \\
 BC & \xrightarrow{g_1} & C
 \end{array}
 \quad
 \begin{array}{ccc}
 CA & \xleftarrow{g_2 1} & CBA \\
 (g \bullet f)_2 \downarrow & & \downarrow 1f_2 \\
 C & \xleftarrow{g_2} & CB
 \end{array}
 \quad (3.1)$$

commute.

For a general  $\mathbb{M}$ -morphism  $g$ , there might be many  $g \bullet f$  making these diagrams commute, but if the maps  $1g_1$  and  $g_2 1$  are epimorphisms, there can be at most one. As far as the existence of  $g \bullet f$ , this will clearly become easier to analyze if  $1g_1$  and  $g_2 1$  are regular epimorphisms. In fact it will turn out that there is a  $g \bullet f$  provided that  $g_1$  and  $g_2$  lie in  $\mathcal{Q}$ , in which case we say that the  $\mathbb{M}$ -morphism  $g$  is *dense*. The key step is the following result.

**Lemma 3.1.** *Let  $f: A \rightharpoonup B$  and  $g: B \rightharpoonup C$  be  $\mathbb{M}$ -morphisms with  $g$  dense and multiplicative; in particular, this includes non-degeneracy of  $C$ . Then for any morphism  $s: X \rightarrow BC$ , the composite*

$$AX \xrightarrow{1s} ABC \xrightarrow{f_1 1} BC \xrightarrow{g_1} C$$

*depends on  $s$  only through  $g_1.s$ . Dually, for any morphism  $s: X \rightarrow CB$ , the composite*

$$XA \xrightarrow{s1} CBA \xrightarrow{1f_2} CB \xrightarrow{g_2} C$$

*depends on  $s$  only through  $g_2.s$ .*

*Proof.* The equal paths around

$$\begin{array}{ccccccc}
 BAX & \xrightarrow{11s} & BABC & \xrightarrow{1f_1 1} & B^2C & \xrightarrow{1g_1} & BC \\
 f_2 1 \downarrow & & \downarrow f_2 11 & & m1 \downarrow & & \downarrow g_1 \\
 BX & \xrightarrow{1s} & B^2C & \xrightarrow{1g_1} & BC & \xrightarrow{g_1} & C
 \end{array}$$

(2.2)      (2.3)

clearly depend only on  $g_1.s$ . Thus the common composite

$$\begin{array}{ccccccc}
 CBAX & \xrightarrow{111s} & CBABC & \xrightarrow{11f_1 1} & CB^2C & \xrightarrow{11g_1} & CBC \xrightarrow{1g_1} C^2 \\
 g_2 11 \downarrow & & & & g_2 1 \downarrow & & \downarrow m \\
 CAX & \xrightarrow{11s} & CABC & \xrightarrow{1f_1 1} & CBC & \xrightarrow{1g_1} & C^2 \xrightarrow{m} C
 \end{array}$$

(2.2)

depends only on  $g_1.s$ . Since  $g_2$  belongs to  $\mathcal{Q}$  so does  $g_2 11$ , and thus the bottom row of this last diagram depends only on  $g_1.s$ . Finally by non-degeneracy of the multiplication of  $C$  we conclude the first claim. The other claim follows symmetrically.  $\square$

**Proposition 3.2.** *If  $f: A \rightharpoonup B$  is an  $\mathbb{M}$ -morphism and  $g: B \rightharpoonup C$  is a dense multiplicative  $\mathbb{M}$ -morphism then there is a unique  $\mathbb{M}$ -morphism  $g \bullet f: A \rightharpoonup C$  making the diagrams (3.1) commute. Furthermore,  $g \bullet f$  is dense or multiplicative if  $f$  is so.*

*Proof.* First, since  $g$  is dense,  $g_1: BC \rightarrow C$  is the coequalizer of maps  $s, s': X \rightarrow BC$ , and this coequalizer is preserved by  $A(-)$ , so that also  $1g_1: ABC \rightarrow AC$  is the coequalizer of  $1s$  and  $1s'$ . By Lemma 3.1, the composites  $g_1.f_1.1s$  and  $g_1.f_1.1s'$  are equal, and so there is a unique map  $(g \bullet f)_1$  making the diagram in (3.1) commute; similarly there is a unique induced  $(g \bullet f)_2$ .

Next we show that  $(g \bullet f)_1$  and  $(g \bullet f)_2$  are the components of an  $\mathbb{M}$ -morphism. To do so, observe that in the commutative diagrams

$$\begin{array}{ccc}
 CBABC & \xrightarrow{g_2 111} & CABC \xrightarrow{11g_1} CAC \\
 1f_2 11 \downarrow & \searrow 11f_1 1 & \downarrow 1f_1 1 \quad (3.1) \quad \downarrow 1(g \bullet f)_1 \\
 CB^2C & \xrightarrow{(2.2)} CB^2C \xrightarrow{g_2 11} CBC \xrightarrow{1g_1} C^2 & \downarrow 1m1 \quad (2.3) \quad \downarrow g_2 1 \quad (2.2) \quad \downarrow m \\
 1m1 \downarrow & \swarrow 1m1 & \downarrow 1m1 \quad (2.3) \quad \downarrow 1g_1 \quad (2.2) \quad \downarrow m \\
 CBC & \xrightarrow{g_2 1} C^2 \xrightarrow{m} C & CBC \xrightarrow{1g_1} C^2 \xrightarrow{m} C
 \end{array}$$

the bottom rows are equal by (2.2), and so the top-right paths are equal. But  $g_2 1g_1$  in the top rows is an epimorphism since  $g$  is dense, so that the right verticals are equal as required.

If  $f$  is dense, then  $f_1 1$ ,  $g_1$ , and  $1g_1$  are all in  $\mathcal{Q}$ , hence so too is  $(g \bullet f)_1$ ; similarly  $(g \bullet f)_2$  is in  $\mathcal{Q}$  and so  $g \bullet f$  is dense.

Finally we show that  $g \bullet f$  is multiplicative if  $f$  is so. In the commutative diagrams

$$\begin{array}{ccc}
 A^2 BC & \xrightarrow{11g_1} & A^2 C \\
 m11 \downarrow & \searrow 1f_1 1 \quad (3.1) & \downarrow 1(g \bullet f)_1 \\
 ABC & \xrightarrow{(2.3)} ABC \xrightarrow{1g_1} AC & \downarrow f_1 1 \quad (3.1) \quad \downarrow (g \bullet f)_1 \\
 & \searrow f_1 1 & \downarrow f_1 1 \quad (3.1) \quad \downarrow (g \bullet f)_1 \\
 & & BC \xrightarrow{g_1} C
 \end{array}$$

the map  $11g_1$  is an epimorphism since  $g$  is dense, and so  $(g \bullet f)_1.1(g \bullet f)_1 = (g \bullet f)_1.m1$  as required.  $\square$

Next we turn to associativity of this composition.

**Proposition 3.3.** *Let  $f: A \rightarrowtail B$ ,  $g: B \rightarrowtail C$ , and  $h: C \rightarrowtail D$  be  $\mathbb{M}$ -morphisms, and suppose that  $g$  and  $h$  are dense and multiplicative. Then  $(h \bullet g) \bullet f = h \bullet (g \bullet f)$ .*

*Proof.* Since  $11h_1: ABCD \rightarrow ABD$  and  $1(h \bullet g)_1: ABD \rightarrow AD$  are epimorphisms, this follows immediately from the commutativity of the following diagrams

$$\begin{array}{ccc}
 ABCD & \xrightarrow{11h_1} & ABD \\
 f_1 11 \downarrow & \searrow f_1 1 & \downarrow 1(h \bullet g)_1 \\
 BCD & \xrightarrow{1h_1} & BD \quad \searrow AD \\
 & \searrow g_1 1 & \downarrow ((h \bullet g) \bullet f)_1 \\
 & & CD \xrightarrow{h_1} D
 \end{array}$$

in which the top left square in the left diagram commutes by functoriality of the monoidal product, and all remaining regions commute by instances of (3.1).  $\square$

As for the identity morphisms, the unital case suggests that the identity  $\mathbb{M}$ -morphism  $i$  on a non-degenerate semigroup  $A$  should have components  $i_1$  and  $i_2$  equal to  $m$ . This does indeed define a multiplicative  $\mathbb{M}$ -morphism by associativity of  $m$ ; it will be dense just when  $m$  lies in  $\mathcal{Q}$ . It follows from the non-degeneracy of  $A$  that  $i: A \rightarrow \mathbb{M}(A)$  is a monomorphism in  $\mathcal{C}$  preserved by the functor  $X(-)$  for any object  $X$ .

**Proposition 3.4.** *Let  $f: A \rightharpoonup B$  be an  $\mathbb{M}$ -morphism.*

- (i) *if  $f$  is dense and multiplicative, then  $f \bullet i = f$ ;*
- (ii) *if  $i$  is dense, then  $i \bullet f = f$ .*

*Proof.* Part (i) follows by commutativity of the diagrams in (2.3) and part (ii) follows by Lemma 2.8.  $\square$

In particular, we now have a category.

**Proposition 3.5.** *There is a category  $\mathcal{M}$ , whose objects are the non-degenerate semigroups with multiplication in  $\mathcal{Q}$ , and whose morphisms are the dense multiplicative  $\mathbb{M}$ -morphisms. The composite  $g \bullet f$  of composable morphisms  $g$  and  $f$  has components as in (3.1), and the identity on an object  $A$  is the  $\mathbb{M}$ -morphism  $i: A \rightharpoonup A$  with components equal to the multiplication  $m$ . The monoidal unit  $I$  equipped with the trivial multiplication is initial in the category  $\mathcal{M}$ .*

*Proof.* The only thing that remains to be proven is that  $I$  is initial. For each object  $A$ , there is an  $\mathbb{M}$ -morphism  $u: I \rightharpoonup A$  with components equal to the identity morphism of  $A$ . This is clearly multiplicative and dense. A general  $\mathbb{M}$ -morphism  $v: I \rightharpoonup A$  will have components given by endomorphisms  $v_1, v_2: A \rightarrow A$  satisfying  $m \cdot v_1 = m \cdot v_2 \cdot 1$ . This will be multiplicative if and only if  $v_1$  and  $v_2$  are idempotent, and it will be dense if and only if  $v_1$  and  $v_2$  are in  $\mathcal{Q}$ ; but the only epimorphic idempotents are the identities.  $\square$

*Remark 3.6.* The proof shows that, in fact, Proposition 3.5 holds also for not necessarily braided monoidal categories  $\mathcal{C}$ .

The category studied by Janssen and Vercruysse in [6] is the case where  $\mathcal{C}$  consists of all modules over a commutative ring, but where we only consider projective modules in defining  $\mathcal{M}$ . The construction of  $\mathcal{M}$  is reminiscent of the Kleisli construction, but does not seem literally to be an example; the obstruction is the need to restrict to dense morphisms.

**Lemma 3.7.** *For a semigroup  $A$  with non-degenerate multiplication  $m$ , the following diagram commutes.*

$$\begin{array}{ccccc}
 \mathbb{M}(A)A & \xrightarrow{1i} & \mathbb{M}(A)^2 & \xleftarrow{i1} & A\mathbb{M}(A) \\
 e_1 \downarrow & & \downarrow m & & \downarrow e_2 \\
 A & \xrightarrow{i} & \mathbb{M}(A) & \xleftarrow{i} & A
 \end{array}$$



*Proof.* We only prove commutativity of the square on the left, symmetric reasoning applies to the other one. In view of Remark 2.7, it is enough to compare the first components of the morphisms around the square.

$$\begin{array}{ccc}
 \mathbb{M}(A)A^2 & \xrightarrow{1i1} & \mathbb{M}(A)^2A \xrightarrow{m1} \mathbb{M}(A)A \\
 \searrow 1m & & \downarrow 1e_1 \quad (2.5) \quad \downarrow e_1 \\
 & & \mathbb{M}(A)A \xrightarrow{e_1} A
 \end{array}
 \quad
 \begin{array}{ccc}
 \mathbb{M}(A)A^2 & \xrightarrow{e_11} & A^2 \xrightarrow{i1} \mathbb{M}(A)A \\
 \downarrow 1m & & \downarrow m \quad (2.1) \quad \downarrow e_1 \\
 \mathbb{M}(A)A & \xrightarrow{e_1} & A
 \end{array}$$

The triangular regions commute since the first component of  $i: A \rightarrow \mathbb{M}(A)$  is the multiplication  $m$ .  $\square$

**Proposition 3.8.** *Consider a closed braided monoidal category  $\mathcal{C}$  and semigroups  $A, B$  in  $\mathcal{C}$ . Assume that the multiplication of  $B$  is non-degenerate and that the pullbacks  $\mathbb{M}(A)$  and  $\mathbb{M}(B)$  in (2.4) exist. Then for any dense and multiplicative morphism  $g: A \rightarrow \mathbb{M}(B)$  there is a unique monoid morphism  $\tilde{g}: \mathbb{M}(A) \rightarrow \mathbb{M}(B)$  obeying  $\tilde{g}.i = g$ .*

*Proof.* Recall from Paragraph 2.9 the multiplicative  $\mathbb{M}$ -morphism  $e: \mathbb{M}(A) \rightarrowtail A$ , and regard  $g$  as a dense multiplicative  $\mathbb{M}$ -morphism  $A \rightarrowtail B$ . The composite  $g \bullet e: \mathbb{M}(A) \rightarrowtail B$  can in turn be regarded as a multiplicative morphism  $\tilde{g}: \mathbb{M}(A) \rightarrow \mathbb{M}(B)$ .

Explicitly, the components of  $\tilde{g}$  are determined by commutativity of the following diagrams.

$$\begin{array}{ccc}
 \mathbb{M}(A)AB & \xrightarrow{1g_1} & \mathbb{M}(A)B \\
 e_11 \downarrow & & \downarrow \tilde{g}_1 \\
 AB & \xrightarrow{g_1} & B
 \end{array}
 \quad
 \begin{array}{ccc}
 BAM(A) & \xrightarrow{g_21} & BM(A) \\
 1e_2 \downarrow & & \downarrow \tilde{g}_2 \\
 BA & \xrightarrow{g_2} & B.
 \end{array}
 \quad (3.2)$$

Since  $1g_1$  is an epimorphism, commutativity of

$$\begin{array}{ccccc}
 A^2B & \xlongequal{\quad} & A^2B & & \\
 1g_1 \downarrow & \searrow i11 & \swarrow m1 & \downarrow 1g_1 & \\
 AB & & \mathbb{M}(A)AB \xrightarrow{e_11} AB & (2.3) & AB \\
 i1 \downarrow & \swarrow 1g_1 & \searrow g_1 & \downarrow g_1 & \\
 \mathbb{M}(A)B & \xrightarrow{\tilde{g}_1} & B & & 
 \end{array}
 \quad (3.2)$$

implies that  $(\tilde{g}.i)_1 = \tilde{g}_1.i1$  is equal to  $g_1$ . Hence it follows by Remark 2.7 that  $\tilde{g}.i = g$ .

Conversely, suppose that  $h: \mathbb{M}(A) \rightarrow \mathbb{M}(B)$  is a multiplicative morphism satisfying  $h.i = g$ ; equivalently,  $h_1.i1 = g_1$ . Then

$$\begin{array}{ccccc}
 & & 1g_1 & & \\
 & \xrightarrow{\quad} & \xrightarrow{\quad} & \xrightarrow{\quad} & \\
 \mathbb{M}(A)AB & \xrightarrow{1i1} & \mathbb{M}(A)^2B & \xrightarrow{1h_1} & \mathbb{M}(A)B \\
 e_11 \downarrow & & \downarrow m1 & & \downarrow h_1 \\
 AB & \xrightarrow{i1} & \mathbb{M}(A)B & \xrightarrow{h_1} & B \\
 & \xrightarrow{\quad} & \xrightarrow{\quad} & & \\
 & & g_1 & & 
 \end{array}$$

commutes by Lemma 3.7 and Proposition 2.10 (ii). By the uniqueness of  $\tilde{g}_1$  rendering commutative the first diagram in (3.2) we conclude that  $h_1 = \tilde{g}_1$  and thus by Remark 2.7 also  $h = \tilde{g}$ .

It remains to see that  $\tilde{g}$  is unital, or equivalently that  $g \bullet u = u$ ; but this follows from the fact that  $I$  is initial in  $\mathcal{M}$ .  $\square$

*Remark 3.9.* We motivated the definition of  $\mathbb{M}$ -morphisms  $f: A \rightharpoonup B$  by the fact that any monoid morphism  $f: A \rightarrow B$  induces such an  $\mathbb{M}$ -morphism with components

$$AB \xrightarrow{f_1} B^2 \xrightarrow{m} B \qquad BA \xrightarrow{1f} B^2 \xrightarrow{m} B \quad (3.3)$$

which render commutative the diagrams in (2.2) and in (2.3). But if  $A$  and  $B$  are merely semigroups and  $f: A \rightarrow B$  multiplicative, then the same definitions still give a multiplicative  $\mathbb{M}$ -morphism  $A \rightharpoonup B$ , which we call  $f^\#$ ; it is just that in this non-unital case the two notions are no longer equivalent. If  $g: B \rightharpoonup C$  is a morphism in  $\mathcal{M}$  and  $f: A \rightarrow B$  a morphism in  $\mathcal{C}$ , then the composite  $g \bullet f^\#$  has components  $(g \bullet f^\#)_1 = g_1 \cdot f_1$  and  $(g \bullet f^\#)_2 = g_2 \cdot 1f$ . On the other hand, if the multiplication of  $B$  is non-degenerate and it belongs to  $\mathcal{Q}$ , and  $f: A \rightarrow B$  is a multiplicative *isomorphism* in  $\mathcal{C}$  and  $g: Z \rightharpoonup A$  an arbitrary  $\mathbb{M}$ -morphism then  $(f^\# \bullet g)_1 = f \cdot g_1 \cdot 1f^{-1}$  and  $(f^\# \bullet g)_2 = f \cdot g_2 \cdot f^{-1}1$ .

We record various facts about the passage from  $f$  to  $f^\#$  in the following proposition.

**Proposition 3.10.** *There is a non-full subcategory  $\mathcal{D}$  of the category of semigroups in  $\mathcal{C}$  whose objects are those semigroups which are non-degenerate and have multiplication in  $\mathcal{Q}$ , and whose morphisms  $f: A \rightarrow B$  are those semigroup morphisms for which the induced  $f^\#$  have components lying in  $\mathcal{Q}$ . There is a faithful functor  $\mathcal{D} \rightarrow \mathcal{M}$  which is the identity on objects and sends  $f$  to  $f^\#$ ; furthermore, this functor is full on isomorphisms.*

*Proof.* The existence of  $\mathcal{D}$  and the faithful functor is evident from the previous discussion. We shall therefore only verify the fact that the functor is full on isomorphisms.

Suppose then that  $f: A \rightharpoonup B$  is an isomorphism in  $\mathcal{M}$ , say with inverse  $g$ . The components  $g_1$  and  $g_2$  of  $g$  lie in  $\mathcal{Q}$ , thus in particular  $g_1$  is the coequalizer of a pair  $w, v: X \rightarrow BAB$  of morphisms in  $\mathcal{C}$ . In the diagram

$$\begin{array}{ccccc} & & B^2 & & \\ & \nearrow f_2 1 & \uparrow (2.2) & \searrow m & \\ XB & \xrightarrow[w1]{v1} BAB & \xrightarrow{1f_1} B^2 & \xrightarrow{m} B & \\ & \searrow g_1 1 & \downarrow f_1 & \nearrow & \\ & & AB & & \end{array} \quad (3.4)$$

the lower region on the right commutes since  $f \bullet g = i$ . Since  $w1$  and  $v1$  agree when composed with the lower path, they agree when composed with the upper path. By non-degeneracy of the multiplication, it follows that  $f_2 \cdot w = f_2 \cdot v$ , and so there is a unique  $f^\flat: A \rightarrow B$  satisfying  $f^\flat \cdot g_1 = f_2$ .

Using (2.2) together with the associativity and the non-degeneracy of the multiplication, commutativity of the lower triangle of (3.4) is seen to be equivalent to the

commutativity of the region marked by  $(*)$  in

$$\begin{array}{ccccc}
 (BA)^2 & \xrightarrow{11g_1} & BA^2 & \xrightarrow{g_11} & A^2 \\
 g_1g_1 \downarrow & \searrow f_211 & \swarrow f_21 & \searrow 1m & \downarrow m \\
 A^2 & & B^2A \xrightarrow{1g_1} BA & & BA \xrightarrow{g_1} A \\
 f^b f^b \downarrow & \swarrow 1f_2 & \searrow f_2 & \swarrow f_2 & \downarrow f^b \\
 B^2 & \xrightarrow{m} & B & \xlongequal{\quad} & B
 \end{array}
 \quad
 \begin{array}{c}
 (2.1) \\
 (2.3) \\
 (*)
 \end{array}$$

Since  $g_1g_1: (BA)^2 \rightarrow A^2$  is epi, commutativity of this proves that  $f^b$  is multiplicative. Using that  $g_11: BAB \rightarrow AB$  is epi, it follows by the commutativity of the square in (3.4) that  $m.f^b1 = f_1$  so that  $(f^b)^\# = f$ .  $\square$

#### 4. MONOIDALITY

In any braided monoidal category, the monoidal product of semigroups  $A$  and  $B$  is again a semigroup with multiplication

$$(AB)^2 \xrightarrow{1c1} A^2B^2 \xrightarrow{mm} AB. \quad (4.1)$$

Our aim is to extend this construction to a monoidal structure on the category  $\mathcal{M}$  of Proposition 3.5. While the category  $\mathcal{M}$  is also available for not necessarily braided monoidal categories  $\mathcal{C}$  (see Remark 3.6), its monoidal structure makes essential use of the braiding of  $\mathcal{C}$ .

**Proposition 4.1.** *If the semigroups  $A$  and  $B$  are non-degenerate, then so is their monoidal product  $AB$ .*

*Proof.* We check that the map sending a morphism  $s: X \rightarrow AB$  to the morphism  $\Phi(s): ABX \rightarrow AB$  given by

$$ABX \xrightarrow{11s} (AB)^2 \xrightarrow{1c1} A^2B^2 \xrightarrow{mm} AB$$

is injective; the other half holds dually. By non-degeneracy of  $A$ , if we know  $\Phi(s)$  then we know the upper, and also the lower, composite in the diagram

$$\begin{array}{ccccc}
 BX & \xrightarrow{1s} & BAB & \xlongequal{\quad} & BAB \xrightarrow{c1} AB^2 \xrightarrow{1m} AB \\
 & & \searrow 1c^{-1} & \nearrow 1c & \\
 & & B^2A & \xrightarrow{m1} & BA \nearrow c
 \end{array}$$

but now by non-degeneracy of  $B$  we know  $c^{-1}.s$  and so in turn we know  $s$  as required.  $\square$

For semigroup morphisms  $f: A \rightarrow B$  and  $f': A' \rightarrow B'$ , also the monoidal product  $ff': AA' \rightarrow BB'$  is compatible with the multiplication (4.1). The components of  $(ff')^\#$  are

$$AA'BB' \xrightarrow{ff'11} (BB')^2 \xrightarrow{1c1} B^2B'^2 \xrightarrow{mm'} BB' \xleftarrow{mm'} B^2B'^2 \xleftarrow{1c1} (BB')^2 \xleftarrow{11ff'} BB'AA',$$

motivating the following construction for more general  $\mathbb{M}$ -morphisms.

**Proposition 4.2.** *If  $f: A \rightarrowtail B$  and  $f': A' \rightarrowtail B'$  are  $\mathbb{M}$ -morphisms, then the pair*

$$AA'BB' \xrightarrow{1c1} ABA'B' \xrightarrow{f_1 f'_1} BB' \qquad BB'AA' \xrightarrow{1c1} BAB'A' \xrightarrow{f_2 f'_2} BB'$$

*defines an  $\mathbb{M}$ -morphism  $AA' \rightarrowtail BB'$ , which is multiplicative or dense if  $f$  and  $f'$  are so.*

*Proof.* The stated morphisms render commutative the diagram of (2.2) by commutativity of

$$\begin{array}{ccccc} BB'AA'BB' & \xrightarrow{111c1} & BB'ABA'B' & \xrightarrow{11f_1 f'_1} & (BB')^2 \\ \downarrow 1c111 & & \downarrow 1c_{B',AB}11 & & \downarrow 1c1 \\ BAB'A'BB' & \xrightarrow{11c_{B',A',B}1} & BABB'A'B' & \xrightarrow{1f_1 1f'_1} & B^2 B'^2 \\ \downarrow f_2 f'_2 11 & & \downarrow f_2 1f'_2 1 & (2.2) & \downarrow mm' \\ (BB')^2 & \xrightarrow{1c1} & B^2 B'^2 & \xrightarrow{mm'} & BB'. \end{array}$$

When it comes to multiplicativity, in view of Remark 2.7, it is enough to check the commutativity of one of the diagrams in (2.3). In the case of the first one, for example, it follows by the commutativity of

$$\begin{array}{ccccc} (AA')^2 BB' & \xrightarrow{111c1} & AA'ABA'B' & \xrightarrow{11f_1 f'_1} & AA'BB' \\ \downarrow 1c111 & & \downarrow 1c_{A',AB}11 & & \downarrow 1c1 \\ A^2 A'^2 BB' & \xrightarrow{11c_{A',A',B}1} & A^2 BA'^2 B' & \xrightarrow{1f_1 1f'_1} & ABA'B' \\ \downarrow mm'11 & & \downarrow m1m'1 & (2.3) & \downarrow f_1 f'_1 \\ AA'BB' & \xrightarrow{1c1} & ABA'B' & \xrightarrow{f_1 f'_1} & BB'. \end{array}$$

Finally if  $f$  and  $f'$  are dense, then  $f_1$  and  $f'_1$  are in  $\mathcal{Q}$ , and so their monoidal product  $f_1 f'_1$  is in  $\mathcal{Q}$ , as is its compsite  $(ff')_1$  with  $1c1$ .  $\square$

**Proposition 4.3.** *The category  $\mathcal{M}$  is monoidal with respect to the usual monoidal product of semigroups, and with monoidal product of morphisms given as in Proposition 4.2.*

*Proof.* The associativity and unit isomorphisms are inherited from  $\mathcal{C}$  as in Remark 3.9. The naturality of these isomorphisms follows from their naturality in  $\mathcal{C}$ , using the description in Remark 3.9 of composition in  $\mathcal{M}$  with  $g^\#$  for an isomorphism  $g$ . It remains only to check that the monoidal product is functorial.

Given morphisms  $f: A \rightharpoonup B$ ,  $g: B \rightharpoonup C$ ,  $f': A' \rightharpoonup B'$ , and  $g': B' \rightharpoonup C'$  in  $\mathcal{M}$ , the right vertical in the diagram

$$\begin{array}{ccccc}
 AA'BB'CC' & \xrightarrow{111c1} & AA'BCB'C' & \xrightarrow{11g_1g'_1} & AA'CC' \\
 \downarrow 1c111 & & \downarrow 1c111 & & \downarrow 1c1 \\
 ABA'B'CC' & \xrightarrow{111c1} & ABA'CB'C' & & \\
 \downarrow f_1f'_111 & & \downarrow 11c11 & & \\
 & & ABCA'B'C' & \xrightarrow{1g_11g'_1} & ACA'C' \\
 & & \downarrow f_11f'_11 & & \downarrow (g \bullet f)_1(g' \bullet f')_1 \\
 BB'CC' & \xrightarrow{1c1} & BCB'C' & \xrightarrow{g_1g'_1} & CC'
 \end{array}$$

is the first component of  $(g \bullet f)(g' \bullet f')$ , but commutativity of the diagram means that it satisfies the defining property of the first component of  $gg' \bullet ff'$ . Thus the monoidal product preserves composition; preservation of identities is straightforward.  $\square$

## 5. MULTIPLIER BIMONOIDS AS COMONOIDS

One of several equivalent ways of describing bimonoids in a braided monoidal category is to say that they are comonoids in the monoidal category of monoids. Our aim is to give an analogous description of (certain) multiplier bimonoids in [4] as comonoids in the monoidal category  $\mathcal{M}$ . This allows us to define morphisms of these multiplier bimonoids as comonoid morphisms.

**Theorem 5.1.** *Let  $\mathcal{C}$  be a braided monoidal category satisfying the standing assumptions of Section 2.1, and let  $\mathcal{M}$  be the induced monoidal category as in Propositions 3.5 and 4.3. For an object  $A$  of  $\mathcal{M}$ , and for morphisms  $t_1, t_2: A^2 \rightarrow A^2$  and  $e: A \rightarrow I$  in  $\mathcal{C}$ , the following assertions are equivalent.*

(i) *There is a comonoid in  $\mathcal{M}$  with counit  $(e: A \rightarrow I \leftarrow A: e)$  and comultiplication*

$$(d_1: A^3 \xrightarrow{1c^{-1}} A^3 \xrightarrow{t_11} A^3 \xrightarrow{1c} A^3 \xrightarrow{m1} A^2 \xleftarrow{1m} A^3 \xleftarrow{c1} A^3 \xleftarrow{1t_2} A^3 \xleftarrow{c^{-1}1} A^3 : d_2).$$

(ii) *There is a multiplier bimonoid  $(A, t_1, t_2, e)$  in  $\mathcal{C}$  such that*

- *the resulting multiplication  $e1.t_1 = 1e.t_2$  is equal to the given one  $m: A^2 \rightarrow A$ ,*
- *the counit  $e$ , and the morphisms  $d_1$  and  $d_2$  in part (i) lie in  $\mathcal{Q}$ .*

*Proof.* Let us spell out what is being asserted in (i). An  $\mathbb{M}$ -morphism  $A \rightharpoonup I$  is just a morphism  $e: A \rightarrow I$  in  $\mathcal{C}$ ; it will be multiplicative as an  $\mathbb{M}$ -morphism if and only if it is multiplicative as a  $\mathcal{C}$ -morphism, in the sense that

$$e.m = ee \tag{5.1}$$

and it will be dense if and only if it lies in  $\mathcal{Q}$ . Using the associativity and the non-degeneracy of the multiplication, we see that the pair  $(d_1, d_2)$  renders commutative (2.2) if and only if

$$m1.1t_1 = 1m.t_21, \tag{5.2}$$

and it renders commutative the first diagram in (2.3), meaning

$$d_1.1d_1 = d_1.m11, \tag{5.3}$$

if and only if the ‘short fusion equation’

$$m1.c^{-1}1.1t_1.c1.1t_1 = t_1.m1 \quad (5.4)$$

holds.

From (5.2) it follows that

$$\begin{array}{ccccc}
 A^4 & \xrightarrow{11m} & A^3 & \xrightarrow{1t_1} & A^3 \\
 \downarrow 1t_1 1 & \searrow t_2 11 & \downarrow t_2 1 & & \downarrow m1 \\
 & A^4 & \xrightarrow{11m} & A^3 & \\
 & \downarrow 1m1 & & & \\
 A^4 & \xrightarrow{m11} & A^3 & & \\
 \downarrow 11m & & \downarrow 1m & & \\
 A^3 & \xrightarrow{m1} & A^2 & & 
 \end{array}
 \quad (5.2)$$

(associativity)

commutes; hence by the non-degeneracy of  $m$ ,

$$t_1.1m = 1m.t_1 1 \quad (5.5)$$

or equivalently

$$d_1.11m = 1m.d_1. \quad (5.6)$$

The  $\mathbb{M}$ -morphism  $e : A \rightarrowtail I$  is a left counit for the comultiplication  $d : A \rightarrowtail A^2$  if and only if either (and hence by Remark 2.7 both) of the diagrams

$$\begin{array}{ccc}
 A^4 & \xrightarrow{1em} & A^2 \\
 \downarrow d_1 1 & & \downarrow m \\
 A^3 & \xrightarrow{em} & A
 \end{array}
 \quad
 \begin{array}{ccccc}
 A^4 & \xrightarrow{c11} & A^4 & \xrightarrow{em1} & A^2 \\
 \downarrow 1d_2 & & \downarrow m & & \downarrow m \\
 A^3 & \xrightarrow{c1} & A^3 & \xrightarrow{em} & A
 \end{array}$$

commutes. Using (5.1), the associativity and the non-degeneracy of  $m$ , and the fact that  $1e1$  is an epimorphism, they are seen to be equivalent to

$$e1.t_1 = m \quad (5.7)$$

and

$$e1.t_2 = e1, \quad (5.8)$$

respectively. Symmetrically,  $e : A \rightarrowtail I$  is a right counit if and only if

$$1e.t_2 = m, \quad (5.9)$$

equivalently,

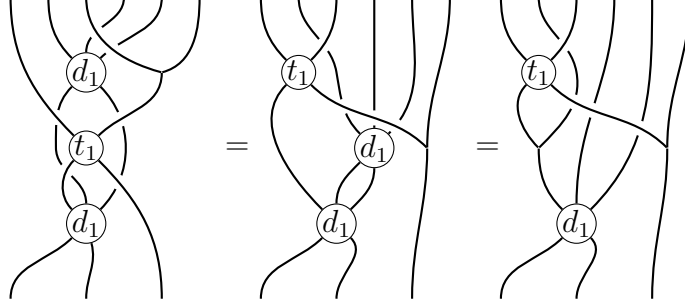
$$1e.t_1 = 1e. \quad (5.10)$$

Finally,  $d : A \rightarrowtail A^2$  is coassociative if and only if the uniquely determined right verticals of the the diagrams

$$\begin{array}{ccccccc}
 A^6 & \xrightarrow{11c11} & A^6 & \xrightarrow{111c1} & A^6 & \xrightarrow{1d_1 m} & A^4 \\
 \downarrow d_1 111 & & & & & & \downarrow \\
 A^5 & \xrightarrow{1c11} & A^5 & \xrightarrow{11c1} & A^5 & \xrightarrow{d_1 m} & A^3
 \end{array}
 \quad
 \begin{array}{ccccccc}
 A^6 & \xrightarrow{11c11} & A^6 & \xrightarrow{1md_1} & A^4 & & \\
 \downarrow d_1 111 & & & & \downarrow & & \\
 A^5 & \xrightarrow{1c11} & A^5 & \xrightarrow{md_1} & A^3 & & 
 \end{array}
 \quad (5.11)$$

are equal to each other. (Clearly there is an equivalent equation involving  $d_2$ .)

In the following string calculation, the first equality holds by (5.5) and the second by (5.3)



and the result is that the unique morphism rendering commutative the first diagram in (5.11) is

$$A^4 \xrightarrow{cc^{-1}} A^4 \xrightarrow{1t_11} A^4 \xrightarrow{c^{-1}c} A^4 \xrightarrow{d_11} A^3. \quad (5.12)$$

Therefore  $d: A \rightrightarrows A^2$  is coassociative if and only if the second diagram in (5.11) commutes, with the morphism (5.12) in the right vertical.

By naturality, coherence, and by the associativity of  $m$ , it follows that

$$\begin{array}{ccccc} A^4 & \xrightarrow{11c} & A^4 & \xrightarrow{1m1} & A^3 \\ d_11 \downarrow & & & & \downarrow d_1 \\ A^3 & \xrightarrow{1c} & A^3 & \xrightarrow{m1} & A^2 \end{array}$$

commutes. Applying this together with (5.5) and using the non-degeneracy of  $m$ , we see that commutativity of the second diagram in (5.11), with (5.12) in the right vertical, is equivalent to the ‘fusion equation’

$$t_11.1c.t_11.1c^{-1}.1t_1 = 1t_1.t_11. \quad (5.13)$$

Summarizing, we proved so far that assertion (i) is equivalent to the validity of (5.1), (5.2), (5.4), (5.7), (5.9), and (5.13).

On the other hand, it was shown in [4, Proposition 3.7] that assertion (ii) is equivalent to the validity of (5.1), (5.7), (5.9), (5.13), and the compatibility condition

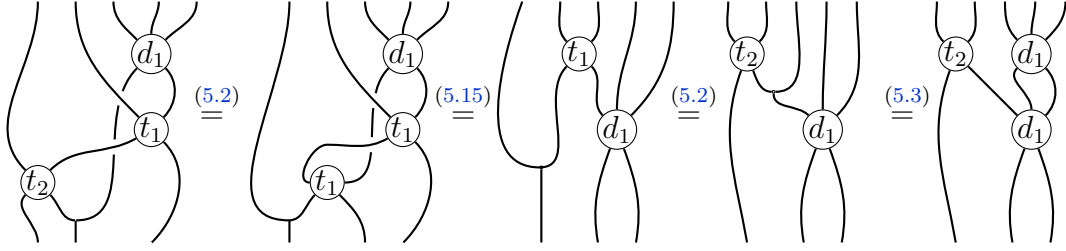
$$t_21.1t_1 = 1t_1.t_21. \quad (5.14)$$

Thus it remains to show that, in the presense of (5.1), (5.7), (5.9), and (5.13), the condition (5.14) is equivalent to the conjunction of (5.2) and (5.4).

For the forward implication, condition (5.2) holds by (3.2) in [4], and (5.4) holds by [4, Remark 3.6]. For the converse, first observe that, by (5.5) and non-degeneracy, the fusion equation (5.13) is equivalent to

(5.15)

and now



and cancelling  $d_1$  from the left and right hand composite and using non-degeneracy now gives the desired result.  $\square$

The result [6, Proposition 3.1] can be seen as the special case where  $\mathcal{C}$  is the category of modules over a commutative ring, and where the object  $A$  is projective over that ring: it then states that if  $A$  is a multiplier Hopf algebra in the sense of Van Daele [7] then it can be seen as a comonoid in the corresponding category  $\mathcal{M}$ . Theorem 5.1 shows that the restriction to projective modules can be avoided, as well as generalizing to other braided monoidal categories.

Let us stress that in Theorem 5.1 we only described *certain* multiplier bimonoids as comonoids in  $\mathcal{M}$  (those whose multiplication is non-degenerate, and for which the multiplication as well as morphisms  $d_1, d_2$  and  $e$  belong to  $\mathcal{Q}$ ). Also, *not every* comonoid in  $\mathcal{M}$  corresponds to a multiplier bimonoid (only those whose morphisms  $d_1, d_2$  have a particular form). Results stronger in both aspects can be achieved by taking a different point of view. Recall that comonoids in a monoidal category  $\mathcal{M}$  can be regarded as simplicial maps from the Catalan simplicial set  $\mathbb{C}$  to the nerve of  $\mathcal{M}^\circ$  (meaning the category with the reverse composition) [5]. In [1] we construct a simplicial set which is not necessarily the nerve of any monoidal category, but for which the simplicial maps from  $\mathbb{C}$  to it can be identified with multiplier bimonoids.

## 6. MORPHISMS

We have seen how to identify (certain) multiplier bimonoids in the braided monoidal category  $\mathcal{C}$  with comonoids in the monoidal category  $\mathcal{M}$ . We shall now investigate morphisms of comonoids.

**6.1. Morphisms between comonoids in  $\mathcal{M}$ .** Suppose that  $(C, d, e)$  and  $(C', d', e')$  are comonoids in  $\mathcal{M}$ . We claim that a morphism of comonoids is then a morphism  $f : C \rightarrow C'$  in  $\mathcal{M}$  whose components render commutative the following diagrams.

$$\begin{array}{ccc}
 CC' \xrightarrow{f_1} C' & & C' C^3 C'^2 \xrightarrow{1d_1 11} C' C^2 C'^2 \xrightarrow{11c_1} C' (CC')^2 \xrightarrow{1f_1 f_1} C'^3 \\
 \searrow ee' & \downarrow 111c_1 & \downarrow d'_1 \\
 & C' C (CC')^2 & C'^3 \xrightarrow{f_2 f_1 f_1} C'^2 \\
 & & \downarrow d'_1
 \end{array} \quad (6.1)$$



There is also of course an equivalent, symmetric set of diagrams with the roles of the components interchanged:

$$\begin{array}{ccc}
 C''C \xrightarrow{f_2} C' & & C'^2C^3C' \xrightarrow{11d_21} C'^2C^2C' \xrightarrow{1c11} (C'C)^2C' \xrightarrow{f_2f_21} C'^3 \\
 \searrow e'e & \downarrow e' & \downarrow 1c111 \\
 I & & (C'C)^2CC' \xrightarrow{f_2f_2f_1} C'^3 \xrightarrow{d'_2} C'^2.
 \end{array}$$

Moreover, using the non-degeneracy of  $C'^2$ , the second diagram of (6.1) is seen to be equivalent also to either of the symmetric diagrams

$$\begin{array}{ccccc}
 C'^3C^3 & \xrightarrow{d'_2d_1} & C'^2C^2 & \xrightarrow{1c1} & (C'C)^2 \\
 \downarrow 11f_211 & & \downarrow f_2f_2 & & \downarrow 11f_111 \\
 C'^3C^2 & \xrightarrow{d'_211} & C'^2C^2 & \xrightarrow{1c1} & (C'C)^2 \xrightarrow{f_2f_2} C'^2
 \end{array}
 \quad
 \begin{array}{ccccc}
 C^3C'^3 & \xrightarrow{d_2d'_1} & C^2C'^2 & \xrightarrow{1c1} & (CC')^2 \\
 \downarrow 11f_111 & & \downarrow f_1f_1 & & \downarrow f_1f_1 \\
 C^2C'^3 & \xrightarrow{11d'_1} & C^2C'^2 & \xrightarrow{1c1} & (CC')^2 \xrightarrow{f_1f_1} C'^2.
 \end{array}$$

We now explain why (6.1) is equivalent to preservation of the comonoid structure. Counitality of  $f$  is clearly equivalent to commutativity of the first diagram of (6.1); and  $f$  is comultiplicative if and only if the uniquely determined right verticals of the diagrams

$$\begin{array}{ccc}
 CC'^3 & \xrightarrow{1d'_1} & CC'^2 \\
 \downarrow f_111 & & \downarrow \\
 C'^3 & \xrightarrow{d'_1} & C'^2
 \end{array}
 \quad
 \begin{array}{ccc}
 C^3C'^2 & \xrightarrow{11c1} & C(CC')^2 \xrightarrow{1f_1f_1} CC'^2 \\
 \downarrow d_111 & & \downarrow \\
 C^2C'^2 & \xrightarrow{1c1} & (CC')^2 \xrightarrow{f_1f_1} C'^2
 \end{array} \tag{6.2}$$

are equal to each other. Let us denote this common morphism by  $g: CC'^2 \rightarrow C'^2$ . Using the non-degeneracy of the multiplication of  $C'^2$  and the fact that  $d'_21111: C'^3CC'^3 \rightarrow C'^2CC'^3$  is an epimorphism, commutativity of the first diagram is equivalent to commutativity of

$$\begin{array}{ccccccc}
 C'^3CC'^3 & \xrightarrow{d'_21111} & C'^2CC'^3 & \xrightarrow{111d'_1} & C'^2CC'^2 & \xrightarrow{11g} & C'^4 \xrightarrow{1c1} C'^4 \\
 \downarrow d'_21111 & & & & & & \downarrow m'm' \\
 C'^2CC'^3 & \xrightarrow{11f_111} & C'^5 & \xrightarrow{11d'_1} & C'^4 & \xrightarrow{1c1} & C'^4 \xrightarrow{m'm'} C'^2
 \end{array}$$

and by (2.2) and the non-degeneracy of  $C'^2$  again, this is further equivalent to commutativity of

$$\begin{array}{ccc}
 C''CC'^3 & \xrightarrow{11d'_1} & C''CC'^2 \xrightarrow{1g} C'^3 \\
 \downarrow 1f_111 & & \downarrow d'_1 \\
 C'^4 & \xrightarrow{1d'_1} & C'^3 \xrightarrow{d'_1} C'^2.
 \end{array}$$

By commutativity of the diagram

$$\begin{array}{ccccc}
 C'CC'^3 & \xrightarrow{1f_111} & C'^4 & \xrightarrow{1d'_1} & C'^3 \\
 \downarrow 11d'_1 & \searrow f_2111 & \downarrow m'_{11} & & \downarrow d'_1 \\
 & & C'^4 & \xrightarrow{m'_{11}} & C'^3 \\
 & & \downarrow 1d'_1 & \searrow d'_1 & \\
 C'CC'^2 & \xrightarrow{f_211} & C'^3 & \xrightarrow{d'_1} & C'^2
 \end{array}
 \quad \begin{array}{l} (2.2) \\ (2.3) \end{array}$$

and the fact that  $11d'_1$  is an epimorphism, commutativity of the first diagram in (6.2) is further equivalent to commutativity of

$$\begin{array}{ccc}
 C'CC'^2 & \xrightarrow{1g} & C'^3 \\
 f_211 \downarrow & & \downarrow d'_1 \\
 C'^3 & \xrightarrow{d'_1} & C'^2
 \end{array} \quad (6.3)$$

We conclude that  $f$  is comultiplicative if and only if the second diagram in (6.2) commutes, having in the right vertical the unique morphism  $g$  rendering commutative (6.3).

By similar steps to those used to analyze the first diagram, commutativity of the second diagram in (6.2) is seen to be equivalent to commutativity of

$$\begin{array}{ccccccc}
 C'CC'^3C'^2 & \xrightarrow{111c1} & C'CC(C'')^2 & \xrightarrow{11f_1f_1} & C'CC'^2 & \xrightarrow{1g} & C'^3 \\
 \downarrow 1d_111 & & & & & & \downarrow d'_1 \\
 C'CC'^2C'^2 & \xrightarrow{11c1} & C'(CC')^2 & \xrightarrow{1f_1f_1} & C'^3 & \xrightarrow{d'_1} & C'^2
 \end{array}$$

or, writing the top right path in an equal form via (6.3), to commutativity of the second diagram in (6.1).

**6.2. Morphisms between multiplier bimonoids.** We may define morphisms between the multiplier bimonoids in part (ii) of Theorem 5.1 as comonoid morphisms between the corresponding comonoids in part (i) of Theorem 5.1. This leads to the following explicit description:

Let  $(A, t_1, t_2, e)$  and  $(A', t'_1, t'_2, e')$  be bimonoids in  $\mathcal{C}$  obeying the conditions in Theorem 5.1 (ii). We claim that a morphism of multiplier bimonoids from  $(A, t_1, t_2, e)$  to  $(A', t'_1, t'_2, e')$  is a morphism  $f : A \rightarrowtail A'$  in the category  $\mathcal{M}$  of Proposition 3.5 whose components render commutative the following diagrams.

$$\begin{array}{ccc}
 A'A & \xrightarrow{f_2} & A' \\
 \searrow e'e & & \downarrow e' \\
 & & I
 \end{array}
 \quad
 \begin{array}{ccccc}
 A'^2A^2 & \xrightarrow{1f_21} & A'^2A & \xrightarrow{t'_21} & A'^2A \\
 t'_2t_1 \downarrow & & & & \downarrow 1f_2 \\
 A'^2A^2 & \xrightarrow{1c1} & (A'A)^2 & \xrightarrow{f_2f_2} & A'^2
 \end{array} \quad (6.4)$$

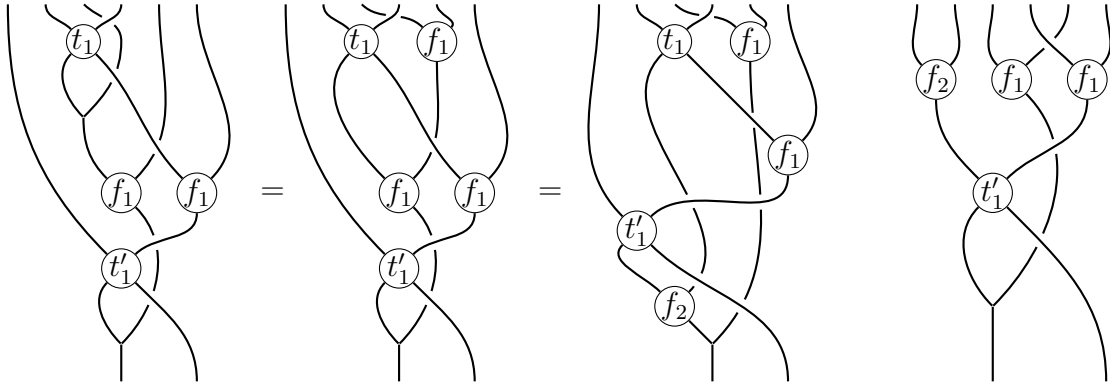
Once again, there is an equivalent, symmetric set of diagrams with the roles of the components interchanged:

$$\begin{array}{ccc}
 AA' & \xrightarrow{f_1} & A' \\
 \searrow ee' & & \downarrow e' \\
 & & I
 \end{array}
 \qquad
 \begin{array}{ccccc}
 A^2 A'^2 & \xrightarrow{1f_1 1} & AA'^2 & \xrightarrow{1t'_1} & AA'^2 \\
 t_2 t'_1 \downarrow & & & & \downarrow f_1 1 \\
 A^2 A'^2 & \xrightarrow{1c 1} & (AA')^2 & \xrightarrow{f_1 f_1} & A'^2
 \end{array}$$

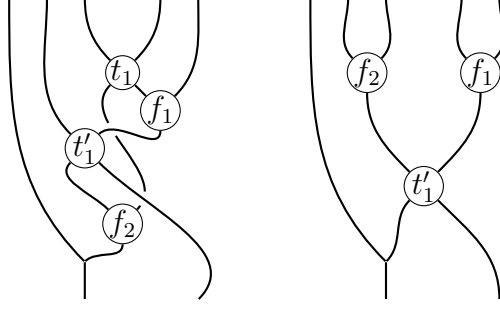
Moreover, using the non-degeneracy of  $A'^2$ , the second diagram of (6.4) is seen to be equivalent also to either of the symmetric diagrams

$$\begin{array}{ccc}
 A' A^2 A' & \xrightarrow{1t_1 1} & A' A^2 A' \xrightarrow{11f_1} A' AA' \\
 \downarrow f_2 f_1 & & \downarrow c 1 \\
 & & AA'^2 \\
 & & \downarrow 1t'_1 \\
 & & AA'^2 \\
 & & \downarrow c^{-1} 1 \\
 & & A' AA' \\
 & & \downarrow f_2 1 \\
 A'^2 & \xrightarrow{t'_1} & A'^2
 \end{array}
 \qquad
 \begin{array}{ccc}
 A' A^2 A' & \xrightarrow{1t_2 1} & A' A^2 A' \xrightarrow{f_2 11} A' AA' \\
 \downarrow f_2 f_1 & & \downarrow 1c \\
 & & A'^2 A \\
 & & \downarrow t'_2 1 \\
 & & A'^2 A \\
 & & \downarrow 1c^{-1} \\
 & & A' AA' \\
 & & \downarrow 1f_1 \\
 A'^2 & \xrightarrow{t'_2} & A'^2
 \end{array}$$

We only need to show that for the particular components  $d_1$  and  $d_2$  in Theorem 5.1 (i), commutativity of the second diagram in (6.1) becomes equivalent to commutativity of the second diagram in (6.4). In terms of strings, this says that the first and last composites below are equal; but since the first three are always equal by (2.3) and (2.2) for  $f$ , this is equivalent to the last two composites being equal.



Since  $f_1$  belongs to  $\mathcal{Q}$  and the multiplication is non-degenerate, this is in turn equivalent to the equality of the following composites.



In the left diagram, use (2.1) for  $f$ , (5.2), and (2.2) for  $f$ ; in the right, use (5.2) and (2.2) for  $f$ . The equality of the resulting composites is equivalent, by non-degeneracy, to commutativity of the second diagram in (6.4).

**Example 6.3.** Let  $A$  and  $A'$  be multiplier bimonoids satisfying the conditions in Theorem 5.1 (ii) and let  $g: A \rightarrow A'$  be a morphism in the category  $\mathcal{D}$  of Proposition 3.10. Then  $g^\#$  is a morphism of multiplier bimonads if and only if  $e'.g = e$  and  $t'_1.gg = gg.t_1$ .

Indeed, the top right path of the first diagram of (6.4) takes the form in any of the equal paths in

$$\begin{array}{ccccc} A'A & \xrightarrow{1g} & A'^2 & \xrightarrow{m'} & A' \\ e'1 \downarrow & & e'e' \downarrow & (5.1) & \downarrow e' \\ A & \xrightarrow{g} & A' & \xrightarrow{e'} & I = I. \end{array}$$

Since  $e'1: A'A \rightarrow A$  is an epimorphism, this is equal to  $e'e$  (in the left bottom path of the first diagram of (6.4)) if and only if  $e'.g = e$ .

The top right path of the second diagram of (6.4) takes the form of any of the equal paths in

$$\begin{array}{ccccccc} A'^2 A^2 & \xrightarrow{11g1} & A'^3 A & \xrightarrow{1m'1} & A'^2 A & \xrightarrow{t'_2 1} & A'^2 A \\ t'_2 11 \downarrow & & 111g \downarrow & & & & \downarrow 11g \\ A'^2 A^2 & & A'^4 & \xrightarrow{1m'1} & A'^3 & \xrightarrow{t'_2 1} & A'^3 \\ 11gg \downarrow & & 11t'_1 \downarrow & & & & \downarrow 1t'_1 (5.2) \\ A'^4 & & A'^4 & & & & \\ & & 11c^{-1} \downarrow & & & & \\ & & A'^4 & \xrightarrow{1t'_1 1} & A'^4 & \xrightarrow{11c} & A'^4 & \xrightarrow{1m'1} & A'^3 & \downarrow 1m' \\ 11t'_1 \downarrow & & t'_2 11 \downarrow & (5.2) & \downarrow m'11 & & & & \downarrow m'1 \\ A'^4 & \xrightarrow{11c^{-1}} & A'^4 & \xrightarrow{1m'1} & A'^3 & \xrightarrow{m'11} & A'^3 & \xrightarrow{(associativity)} & A'^2 \\ 1c1 \downarrow & & & & & & & & \\ A'^4 & \xrightarrow{11m'} & A'^3 & \xrightarrow{m'1} & A'^2. \end{array}$$

Using the form of  $d'_2$  together with the non-degeneracy and the associativity of  $m'$ , this is equal to the composite  $m'm'.1c1.11gg.t'_2t_1$  (occurring in the left bottom path of the first diagram of (6.4)) if and only if

$$m'm'.1c1.11t'_1.11gg.d'_211 = m'm'.1c1.11gg.11t_1.d'_211.$$

Since  $d'_211$  is an epimorphism and  $C'^2$  is non-degenerate, this is equivalent to  $t'_1.gg = gg.t_1$ .

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